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Darboux transformations for a 6-point scheme

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Abstract
We introduce the Jonas transformation (a transformation of the Darboux type) for the general second-order differential equation in two independent variables. We present a discrete version of the transformation for a 6-point difference scheme. The scheme is appropriate for the solution of a hyperbolic-type initial-boundary value problem. We discuss several reductions and specifications of the transformations.

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1. Introduction

One can observe an increasing role of difference equations over the past few decades. Efforts were undertaken to discretize differential equations so as not to lose properties (e.g. symmetries) that differential equations exhibit. It turned out quickly that difference equations in many aspects are richer and more fundamental than their continuous counterparts (many interesting structures disappear under a continuum limit) and difference equations started to be something more than equations mimicking differential equations. In the present work, we encounter essential differences between discrete and continuous mathematical structures once more.

The aim of this paper is to complete the existing theory of the Darboux transformations (more precisely we focus here on generalizations of the Moutard transformation [1, 2] and the Jonas fundamental transformation [3], thus we use the term 'Darboux transformations' in the broadest possible sense that include for example the binary Darboux transformations—cf [4]) for differential equations and what more important to show the impact of the generalization on the theory of the Darboux transformations for difference equations.

The main idea of this paper is to start systematic surveys that can free the theory of integrable systems from their strong dependence on coordinate systems (parametrization of surfaces), desirable by many physicists author spoke to. Due to results of this paper one can compensate lack of possibility of change of independent variables $\tilde{x} = f(x, y)$, $\tilde{y} = g(x, y)$ in the case of difference equations. We expect that this 'compensation' will be especially important in surveys on integrable aspects of the difference geometry [5, 6].

We recall that the classical fundamental transformation by Jonas [3, 7] (regarded as the most general Darboux transformation) acts on the conjugate nets in projective space $\mathbb{P}^n$. Two-dimensional conjugate nets in $\mathbb{P}^n$ are maps $\mathbb{R}^2 \ni (x, y) \mapsto (\psi_1(x, y), \dots, \psi_{n+1}(x, y)) \in \mathbb{R}^{n+1}$ (where sequence $(\psi_1, \dots, \psi_{n+1})$ is interpreted as homogeneous coordinates of $\mathbb{P}^n$) such that $(\psi_1, \dots, \psi_{n+1})$ is $n + 1$-tuple of solutions of scalar equation (from now on, unless otherwise stated, small letters denote functions of real independent variables x and y and subscripts preceded by comma denote partial differentiation with respect to indicated variables)

$$\psi_{,xy} + w\psi_{,x} + z\psi_{,y} + f\psi = 0. \tag{1}$$

Thus in the case when the net is two-dimensional the Jonas transformation provides us with the Darboux transformation for the two-dimensional linear hyperbolic differential equation in a canonical form and the transformation is nothing but the spatial part of the Darboux–Bäcklund transformation for the two-component KP hierarchy.

When the net is more than two-dimensional the Jonas fundamental transformation provides us with the Darboux transformation for the set of compatible equations of the form (1) and serves as the spatial part of the Darboux–Bäcklund transformation for the multicomponent KP hierarchies (in other words yields the Bäcklund transformation for the n-wave interaction equations called sometimes the Darboux equations; see e.g. [8, 9]).

The fundamental transformation has been successfully translated into the discrete language [10–12]. The discrete counterpart of the conjugate nets is the quadrilateral lattices in $\mathbb{P}^n$ governed by a system of equations of the type (unless otherwise stated in almost the whole paper capital letters denote functions of two discrete variables m and n ($(m, n) \in \mathbb{Z}^2$), Δ_m and Δ_n denotes forward difference operators $\Delta_m\Psi := \Psi_{m+1,n} - \Psi$ and $\Delta_n\Psi := \Psi_{m,n+1} - \Psi$ while Δ_{-m} and Δ_{-n} denotes backward difference operators $\Delta_{-m}\Psi := \Psi_{m-1,n} - \Psi$ and $\Delta_{-n}\Psi := \Psi_{m,n-1} - \Psi$, note we identify $\Psi \equiv \Psi_{m,n}$ and in the whole paper we apply this convention):

$$\Delta_m\Delta_n\Psi + A\Delta_m\Psi + B\Delta_n\Psi + C\Psi = 0. \tag{2}$$

That is why in recent years notion of integrability of discrete (difference) equations was often related to planarity—the 4-point schemes were the building blocks of the theory.

The idea to consider equations of integrable systems theory not only on $\mathbb{Z}^2$ lattice but also on more sophisticated quad-graphs, appeared only recently [13–17]. Even so, the 4-point schemes remained the building blocks of the theory.

In the present paper, we show that planarity is not crucial from the point of view of integrable systems. The theory of the Darboux transformations can be extended to more general schemes: a 6-point difference scheme and a 7-point self-adjoint difference scheme.

It turns out that the general second-order differential equation in two independent variables

$$(a\psi_{,x} + c\psi_{,y})_{,x} + (c\psi_{,x} + b\psi_{,y})_{,y} + w\psi_{,x} + z\psi_{,y} - f\psi = 0$$

is covariant under the Darboux transformation (section 3) so the conjugate nets are no longer of key importance. On the discrete level it reflects in the fact that one can generalize the 4-point scheme to the 6-point scheme (see figure 1 and section 4)

$$A\Psi_{m+2,n} + B\Psi_{m,n+2} + 2C\Psi_{m+1,n+1} + G\Psi_{m+1,n} + H\Psi_{m,n+1} = F\Psi.$$

The Moutard transformation for the 7-point self-adjoint scheme (see figure 2)

$$\begin{aligned}\mathcal{A}_{m+1,n}N_{m+1,n} + \mathcal{A}N_{m-1,n} + \mathcal{B}_{m,n+1}N_{m,n+1} + \mathcal{B}N_{m,n-1} + \mathcal{C}_{m+1,n}N_{m+1,n-1}\\ + \mathcal{C}_{m,n+1}N_{m-1,n+1} = \mathcal{F}N\end{aligned}$$

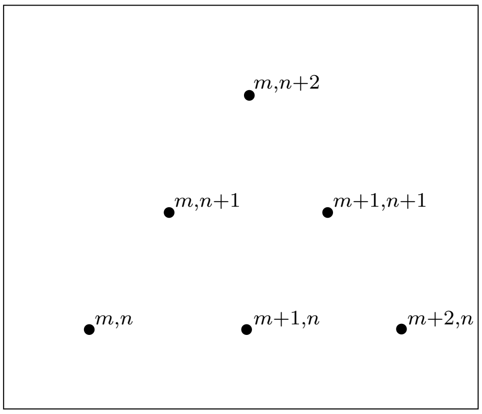

Figure 1. The 6-point scheme.

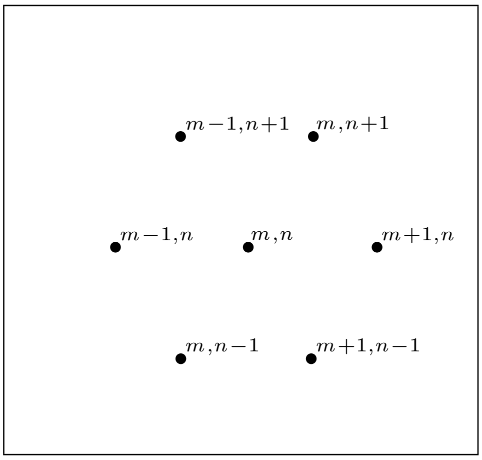

Figure 2. The 7-point scheme.

an example of equation given on a star (cross), has been derived in the paper [18] and that is why we concentrate here mainly on the 6-point scheme. However, the present paper is thought to provide a brief overview on the topic of discretizations of 2D second-order differential equations that are covariant under the Darboux transformations (the reader can find in closing section 7 references to articles on integrable aspects of equations given on stars). At the moment we only underline that a choice of a difference scheme restricts sorts of initial-boundary value problems one can solve by means of the scheme. So it is important to indicate first what sort of initial-boundary conditions one would like to solve and then consider only the schemes that allow to solve the initial-boundary value problem. We start the paper with description of a well-like initial-boundary value problem (section 2) we have in mind while the 6-point scheme is considered.

In this paper we discuss also how the general Darboux transformation can be reduced or specified. We take the stand that introduction of novel terminology (such us ‘specification’) is necessary to discern procedures we deal with. We begin the discussion of reductions and specifications with the continuous case (section 5). First, we consider the Moutard reduction (subsection 5.1) which is a very classical construction [1] but to the best of our knowledge in full generality was given only recently [18]. Second, we discuss specifications (subsection 5.2) and this part is (stands to reason) new, specification to hitherto considered 'conjugate' case or its elliptic counterpart are just examples of such procedure. Third, we introduce convenient gauge specifications (subsection 5.3) the transformations can be written in.

In the discrete case we consider gauge specifications (subsection 6.1) and specifications (subsection 6.2) only, postponing discussion on the Moutard reduction to a forthcoming paper. We would like to mention here that we are not able to show the reduction of the general 6-point scheme that leads to a transformation of the Moutard type. In a forthcoming article we show that the following 10-point scheme

$$\begin{aligned}(\mathcal{A}+\mathcal{B})\Psi &+ (\mathcal{A}_{m+1,n}+\mathcal{B}_{m,n+1})\Psi_{m+1,n+1} + \mathcal{A}\Psi_{m-1,n+1} + \mathcal{B}\Psi_{m+1,n-1} + \mathcal{A}_{m+1,n}\Psi_{m+2,n} \\ &+ \mathcal{B}_{m,n+1}\Psi_{m,n+2} + \mathcal{C}_{m+1,n}\Psi_{m+2,n-1} + \mathcal{C}_{m,n+1}\Psi_{m-1,n+2} + (\mathcal{C}_{m+1,n}-F)\Psi_{m+1,n} \\ &+ (\mathcal{C}_{m,n+1}-F)\Psi_{m,n+1} = 0 \end{aligned} \tag{3}$$

is appropriate for solution of the initial boundary value problem defined in section 2 and can be regarded as a Darboux covariant discretization of the self-adjoint differential equation.

We would like to stress once more that although we deal in the paper with linear equations only, the existence of Darboux transformations makes this paper especially important for the theory of nonlinear integrable systems.

2. Well like initial-boundary value problem for the 6-point scheme

In the present paper we pay special attention to difference schemes that allow us to solve the following initial boundary value problem. We prescribe the function $\Psi(m,n)$ in the following points of the domain (see figure 3)

- initial conditions

$$\{(m,n)\in\mathbb{T}|m+n=0\vee m+n=1\}$$

- boundary conditions

$$\{(m,n)\in\mathbb{T}|(m=s-p_s\wedge n=p_s)\vee(m=s-q_s\wedge n=q_s),\ s=2,3,4,\ldots\}$$

where $\mathbb{T}$ denotes regular triangular lattice, p_s and q_s are functions $\mathbb{N}\backslash\{1\}\ni s\mapsto p_s\in\mathbb{Z}$ such that $\forall s\in\mathbb{N}\backslash\{1\}\, p_s<q_s$.

We concentrate in the paper mainly on the schemes that allow us to find a solution uniquely, at least in the 'upper half-plane' $\{(m,n)\in\mathbb{T}|m+n\geqslant 0\}$ of the lattice.

For instance in the case of the 6-point scheme all the values at white points can be found uniquely provided that two following conditions are satisfied:

(1) $\forall(m,n)\in\mathbb{N}\times\mathbb{N}$, $A_{m,n}\neq 0$, $B_{m,n}\neq 0$, $F_{m,n}\neq 0$;
(2) $\forall s\in\mathbb{N}\backslash\{1\}$, $det[M(s)]\neq 0$; where $M(s)$ are matrices

$$\begin{bmatrix} 2C_{s-p_s-2,p_s} & B_{s-p_s-2,p_s} & 0 & \cdots & & 0 \\ A_{s-p_s-3,p_s+1} & 2C_{s-p_s-3,p_s+1} & B_{s-p_s-3,p_s+1} & 0 & & 0 \\ 0 & & & & \ddots & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & \cdots & 0 & A_{s-q_s+1,q_s-1} & 2C_{s-q_s+1,q_s-1} & B_{s-q_s+1,q_s-1} \\ 0 & & \cdots & 0 & A_{s-q_s,q_s-2} & 2C_{s-q_s,q_s-2} \end{bmatrix}.$$

Similar result can be obtained for the 10-point scheme (3) with the only essential difference that the solution can be found uniquely only in upper half-plane.

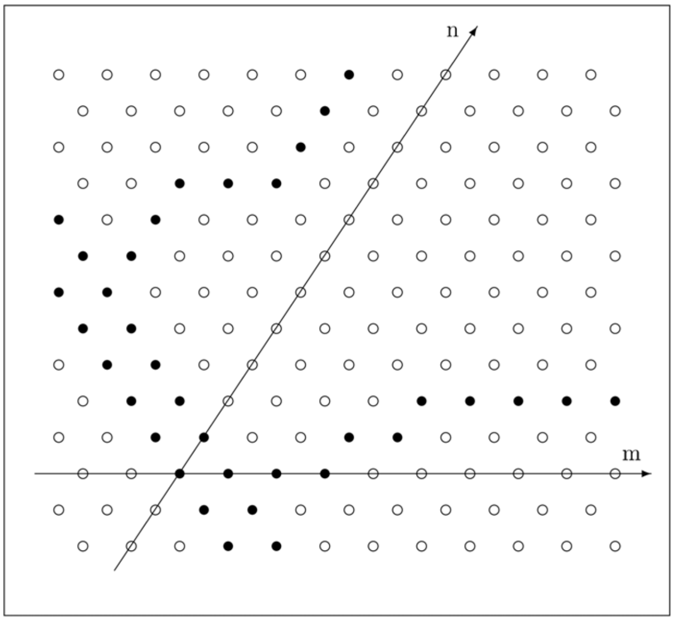

Figure 3. Initial-boundary value problem. The initial values at points of two neighbouring straight lines are given as well as two boundary conditions (black points).

3. Darboux transformations for the 2D second-order differential equation

It is a basic observation that the map $\psi^t \mapsto \bar{\psi}^t$ given by

$$\bar{\psi}^t,_x = \delta\psi^t,_x + \beta\psi^t,_y \qquad \bar{\psi}^t,_y = -\alpha\psi^t,_x - \gamma\psi^t,_y \tag{4}$$

where α, β, γ and δ are $\mathcal{C}^1$ real functions (of independent variables x and y with an open, simply connected subset $\mathcal{D}$ of $\mathbb{R}^2$ as a domain) such that $\forall (x, y) \in \mathcal{D}, \gamma\delta - \alpha\beta \neq 0$ and $\alpha^2 + \beta^2 + (\gamma + \delta)^2 \neq 0$, is an invertible map between solution spaces of two differential equations of second order in two independent variables. Indeed, the compatibility condition of (4), which ensures the existence of $\bar{\psi}^t$ function, reads

$$\mathcal{L}^t\psi^t = 0 \qquad \mathcal{L}^t := \alpha\partial_x^2 + \beta\partial_y^2 + (\gamma + \delta)\partial_x\partial_y + (\alpha,_x + \delta,_y)\partial_x + (\beta,_y + \gamma,_x)\partial_y. \tag{5}$$

Obviously $\bar{\psi}^t$ satisfies an equation of the same type but with barred coefficients:

$$\bar{\alpha} = \frac{\alpha}{\gamma\delta - \alpha\beta}, \qquad \bar{\beta} = \frac{\beta}{\gamma\delta - \alpha\beta}, \qquad \bar{\gamma} = \frac{\delta}{\gamma\delta - \alpha\beta}, \qquad \bar{\delta} = \frac{\gamma}{\gamma\delta - \alpha\beta}. \tag{6}$$

As we shall see, every second-order equation in two independent variables

$$\mathcal{L}^f\psi = 0 \qquad \mathcal{L}^f := a\partial_x^2 + b\partial_y^2 + 2c\partial_x\partial_y + (a,_x + c,_y + w)\partial_x + (b,_y + c,_x + z)\partial_y - f \tag{7}$$

can be transformed into the form (5) through a gauge transformation

$$\mathcal{L}^f \mapsto \mathcal{L} := \hat{\phi}\mathcal{L}^f\hat{\theta} \tag{8}$$

where $\hat{\phi}$ and $\hat{\theta}$ are operators of multiplying by function $\phi(x, y)$ and $\theta(x, y)$ respectively (we tacitly assume the functions are of class $\mathcal{C}^2$). In other words operators (5) and (7) are gauge equivalent, we refer to operator $\mathcal{L}^t$ as *the elementary transformable form* of the second-order differential operator and we can express it in the following theorem.

Theorem 1. *Gauge transformation (8) makes from an arbitrary 2D linear second-order operator $\mathcal{L}^f$ an operator in the elementary transformable form iff*

$$\mathcal{L}^f\theta = 0 \qquad \textit{and} \qquad (\mathcal{L}^f)^\dagger\phi = 0. \tag{9}$$

Proof. Indeed, request that operator $\mathcal{L}$ defined in equation (8) is of the elementary transformable form $\mathcal{L}^t$ (5) gives

$$\alpha = \phi\theta a \qquad \beta = \phi\theta b \qquad \gamma + \delta = 2\phi\theta c \tag{10}$$

$$\mathcal{L}^f\theta = 0 \tag{11}$$

$$\begin{aligned} \alpha,_x + \delta,_y &= [(a\theta),_x + (c\theta),_y + a\theta,_x + c\theta,_y + w\theta]\phi \\ \beta,_y + \gamma,_x &= [(c\theta),_x + (b\theta),_y + c\theta,_x + b\theta,_y + z\theta]\phi. \end{aligned} \tag{12}$$

On introducing the auxiliary function p

$$p := \frac{1}{2\phi\theta}(\delta - \gamma) \tag{13}$$

and treating equations (10) and (13) as the definition of the functions α, β, γ and δ and eliminating these functions from equations (12) one can rewrite equations (12) in the form

$$\begin{aligned} (\theta\phi p),_y &= \phi^2\left[a\left(\frac{\theta}{\phi}\right),_x + c\left(\frac{\theta}{\phi}\right),_y + w\frac{\theta}{\phi}\right] \\ (\theta\phi p),_x &= -\phi^2\left[c\left(\frac{\theta}{\phi}\right),_x + b\left(\frac{\theta}{\phi}\right),_y + z\frac{\theta}{\phi}\right]. \end{aligned} \tag{14}$$

The function p exists provided that

$$\phi\mathcal{L}^f\theta - \theta(\mathcal{L}^f)^\dagger\phi = 0 \tag{15}$$

where $(\mathcal{L}^f)^\dagger$ denotes the operator formally adjoint to the operator $\mathcal{L}^f$

$$(\mathcal{L}^f)^\dagger := \partial_x(a\partial_x + c\partial_y - w) + \partial_y(b\partial_y + c\partial_x - z) - f. \tag{16}$$

Taking into account equations (11) and (15) we get conditions (9).

Conversely, having taken functions θ and ϕ that obey conditions (9) and making gauge transformation (8) one can easily check that operator $\mathcal{L}$ is in the elementary transformable form. First, we check up that the coefficients $\tilde{w}, \tilde{z}$ and $\tilde{f}$ of the operator $\mathcal{L}$ obey $\tilde{f} = 0$ and $\tilde{w},_x + \tilde{z},_y = 0$ due to conditions (9). Finally, due to equation $\tilde{w},_x + \tilde{z},_y = 0$ we can introduce potential κ such that $\tilde{w} = \kappa,_y$ and $\tilde{z} = \kappa,_x$, and give elementary transformable form to the equation $\mathcal{L}\tilde{\psi} = 0$ by means of Leibniz's chain rule. □

It turns out that considerations just presented lead to Darboux transformations for the 2D linear second-order operator $\mathcal{L}^f$. Namely, combination of gauge transformation (8) (with functions ϕ and θ obeying (9)), map (4) ($\psi^t \mapsto \bar{\psi}^t$) and a gauge transformation $\bar{\mathcal{L}}^t \mapsto \bar{\mathcal{L}} := \hat{r}\bar{\mathcal{L}}^t\hat{s}$ is nothing but Darboux transformations for the operator $\mathcal{L}^f$ and we give details in the following conclusion.

Conclusion 1 (Darboux transformations)**.** *We assume that θ and ϕ are $\mathcal{C}^2$ class functions satisfying conditions (9), function p is given by formulae (14), r and s are arbitrary (of class $\mathcal{C}^2$) functions, function d is given by $d := (p^2 - c^2 + ab)\phi\theta$ and obeys the condition $\forall(x, y) \in \mathcal{D}, d \neq 0$. Then the map $\psi \mapsto \bar{\psi}$ given by*

$$\begin{bmatrix} (s\bar{\psi}),_x \\ (s\bar{\psi}),_y \end{bmatrix} = \phi\theta \begin{bmatrix} p + c & b \\ -a & p - c \end{bmatrix} \begin{bmatrix} \left(\frac{\psi}{\theta}\right),_x \\ \left(\frac{\psi}{\theta}\right),_y \end{bmatrix} \tag{17}$$

takes the solution space of equation (7) to the solution space of an equation of the same form but with the new (barred) coefficients:

$$\bar{\mathcal{L}}\bar{\psi} = 0 \qquad \bar{\mathcal{L}} := \bar{a}\partial_x^2 + \bar{b}\partial_y^2 + 2\bar{c}\partial_x\partial_y + (\bar{a}_{,x} + \bar{c}_{,y} + \bar{w})\partial_x + (\bar{c}_{,x} + \bar{b}_{,y} + \bar{z})\partial_y - \bar{f} \tag{18}$$

where the coefficients of equation (18) are related to coefficients of equation (7) by

$$\bar{a} = \frac{asr}{d} \qquad \bar{b} = \frac{bsr}{d} \qquad \bar{c} = \frac{csr}{d}$$

$$\bar{w} = \left[\frac{a}{d}\left(\frac{s}{r}\right)_{,x} + \frac{c}{d}\left(\frac{s}{r}\right)_{,y} - \left(\frac{p}{d}\right)_{,y}\frac{s}{r}\right]r^2 \qquad \bar{z} = \left[\frac{b}{d}\left(\frac{s}{r}\right)_{,y} + \frac{c}{d}\left(\frac{s}{r}\right)_{,x} + \left(\frac{p}{d}\right)_{,x}\frac{s}{r}\right]r^2$$

$$\bar{f} = \left\{-\left[\frac{a}{d}s_x + \frac{c+p}{d}s_y\right]_x - \left[\frac{b}{d}s_y + \frac{c-p}{d}s_x\right]_y\right\}r. \tag{19}$$

4. The 6-point scheme and its Darboux transformations

One can repeat considerations from the previous section in the discrete case. Indeed, starting from pair of equations

$$\Delta_m\bar{\Psi}^t = \delta\Delta_m\Psi^t + \beta\Delta_n\Psi^t \qquad \Delta_n\bar{\Psi}^t = -\alpha\Delta_m\Psi^t - \gamma\Delta_n\Psi^t \tag{20}$$

(where the functions α, β, γ and δ are functions of discrete variables m and n) and writing down their compatibility condition

$$\begin{aligned} &L^t\Psi^t = 0 \\ &L^t := \alpha_{m+1,n}T_mT_m + \beta_{m,n+1}T_nT_n + (\gamma_{m+1,n} + \delta_{m,n+1})T_mT_n \\ &\qquad - (\alpha_{m+1,n} + \alpha + \gamma_{m+1,n} + \delta)T_m - (\beta_{m,n+1} + \beta + \gamma + \delta_{m,n+1})T_n + (\alpha + \beta + \gamma + \delta) \end{aligned} \tag{21}$$

(where T_m and T_n are forward shift operators in m and n direction respectively i.e. $T_mf_{m,n} := f_{m+1,n}$ and $T_nf_{m,n} := f_{m,n+1}$) we find a 6-point scheme. One can ask if it is possible to transform the general 6-point scheme:

$$L^f\Psi = 0 \qquad L^f := AT_mT_m + BT_nT_n + 2CT_mT_n + GT_m + HT_n - F \tag{22}$$

into the form (21) via a gauge transformation

$$L^f \mapsto L := \hat{\Phi}L^f\hat{\Theta} \tag{23}$$

only. The answer is positive. Operators L^t (21) and L^f (22) are gauge equivalent and we refer to the operators L^t as the 6-point operator in *the elementary transformable form*. Through analogy with the continuous case we have the following theorem.

Theorem 2. *Gauge transformation (23) makes from operator L^f (22) an operator in the elementary transformable form L^t (21) iff the function Θ satisfies equation (22)*

$$A\Theta_{m+2,n} + B\Theta_{m,n+2} + 2C\Theta_{m+1,n+1} + G\Theta_{m+1,n} + H\Theta_{m,n+1} = F\Theta \tag{24}$$

and function Φ is a solution of the equation formally adjoint to equation (22)

$$\begin{aligned} &L^{f\dagger}\Phi = 0 \\ &L^{f\dagger} := A_{m-2,n}T_{-m}T_{-m} + B_{m,n-2}T_{-n}T_{-n} + 2C_{m-1,n-1}T_{-m}T_{-n} \\ &\qquad + G_{m-1,n}T_{-m} + H_{m,n-1}T_{-n} - F \end{aligned} \tag{25}$$

where T_{-m} and T_{-n} are backward shift operators in m and n direction respectively i.e. $T_{-m} f_{m,n} := f_{m-1,n}$ and $T_{-n} f_{m,n} := f_{m,n-1}$. Then the functions α, β, γ and δ in equation (21) are given by

$$\begin{aligned} &\alpha = A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} && \beta = B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} \\ &\gamma = (C_{m-1,n} - P_{m-1,n})\Phi_{m-1,n}\Theta_{m,n+1} && \delta = (C_{m,n-1} + P_{m,n-1})\Phi_{m,n-1}\Theta_{m+1,n} \end{aligned} \tag{26}$$

where P is an auxiliary function defined by

$$\begin{aligned} \Delta_{-m}(\Phi\Theta_{m+1,n+1}P) &= -(B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} + B\Phi\Theta_{m,n+2} + C_{m-1,n}\Phi_{m-1,n}\Theta_{m,n+1} \\ &\quad + C\Phi\Theta_{m+1,n+1} + H\Phi\Theta_{m,n+1}) \\ \Delta_{-n}(\Phi\Theta_{m+1,n+1}P) &= A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} + A\Phi\Theta_{m+2,n} + C_{m,n-1}\Phi_{m,n-1}\Theta_{m+1,n} \\ &\quad + C\Phi\Theta_{m+1,n+1} + G\Phi\Theta_{m+1,n}. \end{aligned} \tag{27}$$

The proof is similar to the proof in the continuous case and therefore we omit it. Just like in the continuous case, a combination of gauge transformation (23) (with functions Θ and Φ obeying (24) and (25) respectively), map (20) ($\Psi^t \mapsto \bar{\Psi}^t$) and a gauge transformation $\bar{\mathcal{L}}^t \mapsto \bar{\mathcal{L}} := \hat{R}\bar{\mathcal{L}}^t\hat{S}$ yields the Darboux transformations for the 6-point scheme (22). We give details of the Darboux transformations in the following conclusion.

Conclusion 2 (Darboux transformations for the 6-point scheme)**.** *The map $\Psi \mapsto \bar{\Psi}$ given by*

$$\begin{aligned} \begin{bmatrix} \Delta_m(S\bar{\Psi}) \\ \Delta_n(S\bar{\Psi}) \end{bmatrix} &= \begin{bmatrix} (C_{m,n-1} + P_{m,n-1})\Phi_{m,n-1}\Theta_{m+1,n} & B_{m,n-1}\Phi_{m,n-1}\Theta_{m,n+1} \\ -A_{m-1,n}\Phi_{m-1,n}\Theta_{m+1,n} & (P_{m-1,n} - C_{m-1,n})\Phi_{m-1,n}\Theta_{m,n+1}) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \Delta_m\left(\frac{\Psi}{\Theta}\right) \\ \Delta_n\left(\frac{\Psi}{\Theta}\right) \end{bmatrix} \end{aligned} \tag{28}$$

where function Θ satisfies equation (22) while function Φ is a solution of equation (25) and P is defined via equations (27), takes the solution space of equation (22) to the solution space of an equation of the same form with new (barred) coefficients related to the old ones via

$$\begin{aligned} &\bar{A} = \frac{RS_{m+2,n}\Phi\Theta_{m+2,n}}{D_{m+1,n}}A \qquad \bar{B} = \frac{RS_{m,n+2}\Phi\Theta_{m,n+2}}{D_{m,n+1}}B \qquad \bar{F} = \frac{RS\Phi\Theta}{D}F \\ &\frac{2\bar{C}}{RS_{m+1,n+1}} = \frac{\Theta_{m+2,n}\Phi_{m+1,n-1}(C+P)_{m+1,n-1}}{D_{m+1,n}} + \frac{\Theta_{m,n+2}\Phi_{m-1,n+1}(C-P)_{m-1,n+1}}{D_{m,n+1}} \\ &\frac{\bar{G}}{RS_{m+1,n}} = -\frac{\Theta_{m+2,n}\Phi_{m+1,n-1}(C+P)_{m+1,n-1} + \Theta_{m+2,n}\Phi A}{D_{m+1,n}} \\ &\qquad\qquad - \frac{\Theta_{m,n+1}\Phi_{m-1,n}(C-P)_{m-1,n} + \Theta_{m+1,n}\Phi_{m-1,n}A_{m-1,n}}{D} \\ &\frac{\bar{H}}{RS_{m,n+1}} = -\frac{\Theta_{m,n+2}\Phi_{m-1,n+1}(C-P)_{m-1,n+1} + \Theta_{m,n+2}\Phi B}{D_{m,n+1}} \\ &\qquad\qquad - \frac{\Theta_{m+1,n}\Phi_{m,n-1}(C+P)_{m,n-1} + \Theta_{m,n+1}\Phi_{m,n-1}B_{m,n-1}}{D} \end{aligned} \tag{29}$$

where R and S are arbitrary non-vanishing functions, while D is a function given by $D = [(P_{m-1,n} - C_{m-1,n})(P_{m,n-1} + C_{m,n-1}) + A_{m-1,n}B_{m,n-1}]\Theta_{m+1,n}\Theta_{m,n+1}\Phi_{m-1,n}\Phi_{m,n-1}$ and is assumed not to vanish on the whole lattice.

5. Gauge specifications, specifications and reductions

One can reduce (or gauge, or specify) the Darboux transformations so as they map between the solution spaces of a restricted class of equations. We would like to discern three procedures: reductions, specifications and gauge specifications, for their role in the theory of integrable systems is completely different. We deal with gauge equivalent classes of equations rather than single equation and from this point of view the gauge specifications are not important—they choose a representative of the equivalence class. We mimic change of the independent variables by the specifications. Finally and most importantly, we deal with the reductions which essentially change the transformation (one has to impose constraints on transformation data as we shall see in the example of the Moutard reduction).

It is natural to investigate *linear* constraints first. Therefore we assume that matrix coefficients in equation (17) obey a linear constraint

$$a^{11}(p+c)+a^{12}b-a^{21}a+a^{22}(p-c)=0 \tag{30}$$

where a^{ij} are prescribed functions of x and y. We want the constraint to be preserved, which means the coefficients of the inverse matrix have to obey the same constraint

$$a^{11}(p-c)-a^{12}b+a^{21}a+a^{22}(p+c)=0. \tag{31}$$

From equations (30) and (31) we infer

$$(a^{11}+a^{22})p=0$$

and we shall discuss two cases $p=0$ and $a^{11}+a^{22}=0$ separately.

5.1. *Moutard reduction* $(p=0)$*, reductions*

We have to satisfy the constraints $p=0$ and $(a^{11}-a^{22})c+a^{12}b-a^{21}a=0$. The latter constraint can be satisfied if one take $a^{11}=a^{22}$, $a^{12}=0$ and $a^{21}=0$. To satisfy equations (14) in the presence of condition $p=0$ it is enough to put $\phi=\theta$ and $w=0=z$ (i.e. demand that operator is formally self-adjoint). Moreover the functions r and s are no longer arbitrary, they must obey the constraint $\frac{r}{s}=\text{const}$. As a result we obtain a transformation for formally self-adjoint equations which are usually referred to (in the case $a=0=b$, $c=\frac{1}{2}$, $s=r=\frac{1}{2}\theta$) as the Moutard transformation.

The procedure that imposes the constraints on the transformation data ϕ and θ ($\theta=\phi$ in the Moutard case) we call reduction of the transformation.

5.2. *Specifications* $a^{11}+a^{22}=0$

The reduction is not the only procedure we have at our disposal. If we take $a^{11}=-a^{22}$ we have to satisfy the constraint $2a^{11}c+a^{12}b-a^{21}a=0$. Two examples are as follows:

(a) $a=0=b$, $a^{11}=0$ and $c=1$ which is nothing but specification to the ‘conjugate’ case;
(b) $c=0$, $a^{12}=0=a^{21}$ (with the option for further specification $a=\pm b$).

In these cases we only specify (specialize) the operator and do not affect the transformation data.

5.3. *Gauge specifications, affine form, elementary transformable form*

The idea not to consider the operator itself, but their equivalence classes with respect to gauge transformations goes back to Laplace and Darboux [19]. One can then develop the theory in gauge-independent language, or choose a gauge appropriate to one's needs. We concentrate on the latter procedure. We take two arbitrary functions θ^0, ϕ^0 of $\mathcal{C}^2$ class. Then the operator

$$L^g := \hat{\phi}^0 L^f \hat{\theta}^0$$

has coefficients

$$(a^g, b^g, c^g, f^g) = \theta^0\phi^0(a, b, c, L^f\theta^0)$$
$$w^g = \theta^0\phi^0 w - (\theta^0)^2\left(\frac{\phi^0}{\theta^0}\right)_{,x} a - (\theta^0)^2\left(\frac{\phi^0}{\theta^0}\right)_{,y} c$$
$$z^g = \theta^0\phi^0 z - (\theta_0)^2\left(\frac{\phi^0}{\theta^0}\right)_{,y} b - (\theta_0)^2\left(\frac{\phi^0}{\theta^0}\right)_{,x} c$$

and the operator adjoint to L^g is

$$(L^g)^\dagger := \hat{\theta}^0 L^f \hat{\phi}^0.$$

If in addition we define

$$\psi^g = \frac{\psi}{\theta_0} \qquad \phi^g = \frac{\phi}{\phi_0} \qquad \theta^g = \frac{\theta}{\theta_0}$$

then the form of the transformation remains unaltered. The affine gauge (i.e. a gauge such that $L^f\theta^0 = 0$ and as a consequence $f^g = 0$) is commonly used. Here we would like to draw attention to *the elementary transformable gauge* defined by

$$L^f\theta^0 = 0 \qquad (L^f)^\dagger\phi^0 = 0.$$

This gauge specification brings both the operator L^g and its adjoint $(L^g)^\dagger$ to elementary transformable form i.e., as we know from theorem 1, conditions

$$f^g = 0 \qquad w^g{}_{,x} + z^g{}_{,y} = 0$$

hold. The functions s^g and r^g are no longer arbitrary. To ensure the transformed (barred) equation is in elementary transformable form it is enough (but not necessary) to put the functions s^g and r^g constant.

6. Specifications, discrete case

6.1. Gauge specifications

In the discrete case one can specify the gauge as well. Namely, we take two arbitrary functions $\Theta^0\Phi^0$, then operator

$$L^g := \hat{\Phi}^0 L^f \hat{\Theta}^0$$

has coefficients

$$(A^g, B^g, C^g, G^g, H^g, F^g) = \Phi^0\big(A\Theta^0_{m+2,n}, B\Theta^0_{m,n+2}, C\Theta^0_{m+1,n+1}, G\Theta^0_{m+1,n}, H\Theta^0_{m,n+1}, F\Theta^0\big).$$

If we take Θ^0 such that $L^f\Theta^0 = 0$ we obtain the affine gauge. Then coefficients of operator L^g obey constraint

$$A^g + B^g + 2C^g + G^g + H^g - F^g = 0. \tag{32}$$

If one puts $S = \text{const}$ then the above constraint is preserved under the Darboux transformation. If in addition we take Φ^0 such that $(L^f)^\dagger\Phi^0 = 0$ satisfies the equation then the coefficients of the operator L^g obey also the constraint

$$A^g_{m-1,n+1} + B^g_{m+1,n-1} + 2C^g + G^g_{m,n+1} + H^g_{m+1,n} - F^g_{m+1,n+1} = 0 \tag{33}$$

and the operator L^g is in elementary transformable form of the 6-point scheme. To ensure that both constraints (32) and (33) are preserved under the Darboux transformation it is enough, (but not necessary) to put $R = \text{const}$ and $S = \text{const}$.

6.2. Specifications, quadrilateral lattices and a 3-point scheme

A glance at the transformation laws of the coefficients of the 6-point scheme (29) provides us with conclusions.

(A) All constraints $A = 0$, $B = 0$ and $F = 0$ are preserved under the Darboux transformation (28). Constraint $A = 0 = B$ (or alternatively $A = 0 = F$ or $B = 0 = F$) is a specification to the celebrated 4-point scheme i.e. to quadrilateral lattice case subject of study in many papers (we confine ourselves to citing articles where Darboux transformations are considered):

- quadrilateral lattices (Jonas fundamental transformations) [11, 12, 21–23];
- circular lattices and quadratic reductions (Ribaucour type transformations) [26–29];
- Moutard-type transformations [24, 25];
- symmetric (Goursat) type transformations [30, 31].

The initial boundary value problem suitable for this scheme is no longer of the type mentioned in section 2.

(B) Constraint $C = 0$ is not preserved under the Darboux transformation (28). So we have not got a discretization of specification (b) from section 5.2.

(C) If we impose $F = 0$ together with $A = 0 = B$ we obtain Darboux transformation for a 3-point scheme which corresponds to the continuous degenerate case $a^2 + b^2 + c^2 = 0$.

7. Conclusions and perspectives

In this paper, we have presented the 6-point difference scheme—a Darboux covariant generalization of the extensively studied 4-point schemes. It is worth looking at the literature on integrable aspects of the star-like systems to see that the theory of integrable difference systems based on difference schemes other than the 4-point schemes is still in its infancy.

Indeed, it is remarkable that star-like (or cross-like) operators such as the 7-point scheme or the 5-point scheme appeared in the integrable literature only occasionally [33, 34, 37–40]. Almost none of the pioneering results were used to obtain solutions of nonlinear integrable systems. The only exceptions are works concerning the discrete time Toda equation and its generalizations [41–46], which are nonlinear 5-point schemes themselves and the work [18] the result of which were used to obtain solutions of a generalization of the Toda chain to a two-dimensional lattice [47]. It is the right place to mention that the relationship between integrable systems on quad-graphs and equations on star-like schemes of the discrete time Toda type has been established [13–17] and we would like to refer to this relationship as the sub-lattice approach. In some cases the approach allows one to transfer solutions from an equation on a quad-graph to a star-like equation on a sub-graph (sub-lattice) of the original quad-graph. It is not clear under what circumstances the sub-lattice approach preserves integrability. However in the paper [48] it was proved that for discrete lattices governed by the discrete Moutard equation

$$\Psi_{m+1,n+1} + \Psi = G(\Psi_{m+1,n} + \Psi_{m,n+1}) \tag{34}$$

integrability features like the existence of the Darboux transformations are inherited by the sub-lattice.

We conclude this paper with a list of open problems and perspectives. These are as follows:

- first and the most important, to derive hierarchies of nonlinear equations associated with all the equations presented here;

- second, to generalize the results of this paper to more dimensions;
- third, to generalize the quadratic and symmetric reductions to the 6-point scheme;
- fourth, since the 6-point scheme admits the decomposition $[(M_1T_m + N_1T_n + X_1)(M_2T_m + N_2T_n + X_2) + H]\psi = 0$, it can be used to develop the theory of the Laplace transformations for difference equations [25, 32–35, 40];
- fifth, to develop a simple idea by professor Decio Levi that not only the operators $\Delta_m\Delta_n\Delta_{-m}\Delta_{-n}$ are of importance. For example one can use $T_m - T_{-m}$, $T_n - T_{-n}$, $T_n - T_m$ and $T_nT_m - 1$ operators instead;
- sixth, to investigate the role of the transformations given here in the difference geometry.

We also would like to mention that on this level of generality, giving q-difference analogues of the discrete schemes we have introduced is straightforward [36].

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